Examen 3 (solutions)
201-NYC Algèbre linéaire
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Question 1 [10 points]

a) $\mathbf{A}+\mathbf{C}=\begin{pmatrix} -2 & 7 \\ 3 & -4 \end{pmatrix}$ b) $\mathbf{AB}=\begin{pmatrix} -7 & 16 & 3 \\ -11 & 1 & 14 \end{pmatrix}$ c) $3\mathbf{C}-2\mathbf{BA}=\nexists$

Question 2 [10 points]
a)

$$\left(\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 3 & 0 & 2 & 0 & 1 & 0 \\ -1 & 5 & -1 & 0 & 0 & 1 \end{array}\right) \xrightarrow[L_3\to L_3+L_1]{L_2\to L_2-3L_1} \left(\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & -6 & 5 & -3 & 1 & 0 \\ 0 & 7 & -2 & 1 & 0 & 1 \end{array}\right) \xrightarrow{L_2\to L_2+L_3} \left(\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & 3 & -2 & 1 & 1 \\ 0 & 7 & -2 & 1 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{L_3\to L_3-7L_2} \left(\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & 3 & -2 & 1 & 1 \\ 0 & 0 & -23 & 15 & -7 & -6 \end{array}\right) \xrightarrow{L_3\to -\frac{1}{23}L_3} \left(\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & 3 & -2 & 1 & 1 \\ 0 & 0 & 1 & -\frac{15}{23} & \frac{7}{23} & \frac{6}{23} \end{array}\right)$$

$$\xrightarrow{L_2\to L_2-3L_3} \left(\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -\frac{1}{23} & \frac{2}{23} & \frac{5}{23} \\ 0 & 0 & 1 & -\frac{15}{23} & \frac{7}{23} & \frac{6}{23} \end{array}\right) \xrightarrow{L_1\to L_1+L_3} \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & \frac{8}{23} & \frac{7}{23} & \frac{6}{23} \\ 0 & 1 & 0 & -\frac{1}{23} & \frac{2}{23} & \frac{5}{23} \\ 0 & 0 & 1 & -\frac{15}{23} & \frac{7}{23} & \frac{6}{23} \end{array}\right)$$

$$\xrightarrow{L_1\to L_1-2L_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{10}{23} & \frac{3}{23} & -\frac{4}{23} \\ 0 & 1 & 0 & -\frac{1}{23} & \frac{2}{23} & \frac{5}{23} \\ 0 & 0 & 1 & -\frac{15}{23} & \frac{7}{23} & \frac{6}{23} \end{array}\right)$$

b) $\mathbf{A}^{-1}=-\frac{1}{23}\begin{pmatrix} -10 & 1 & 15 \\ -3 & -2 & -7 \\ 4 & -5 & -6 \end{pmatrix}^T=\begin{pmatrix} \frac{10}{23} & \frac{3}{23} & -\frac{4}{23} \\ -\frac{1}{23} & \frac{2}{23} & \frac{5}{23} \\ -\frac{15}{23} & \frac{7}{23} & \frac{6}{23} \end{pmatrix}$

Question 3 [10 points]

$$\begin{vmatrix} 2 & -3 & k \\ k & -2 & -k \\ 3 & -1 & -1 \end{vmatrix} = 2(2-k)+3(2k)+k(-k+6) = -k^2+10k+4 \overset{?}{=} 0$$

$$k \neq 10 \pm \sqrt{100+16}$$

Question 4 [10 points]

$$\det((-3\mathbf{C})(\mathbf{A}^T\mathbf{B})^{-1}) = \frac{\det(-3\mathbf{C})}{\det(\mathbf{A}^T\mathbf{B})} = \frac{(-3)^4\det(\mathbf{C})}{\det(\mathbf{A}^T)\det(\mathbf{B})} = \frac{(-3)^4\det(\mathbf{C})}{\det(\mathbf{A})\det(\mathbf{B})} = \frac{(-3)^4(5)}{(6)(3)} = \frac{45}{2}$$

Question 5 [10 points]

Non car $f\begin{pmatrix} kx \\ ky \end{pmatrix} = \begin{pmatrix} 3kx - ky \\ 7ky + 2kx - 1 \end{pmatrix} \neq \begin{pmatrix} 3kx - ky \\ 7ky + 2kx - k \end{pmatrix} = kf\begin{pmatrix} x \\ y \end{pmatrix}$

Question 6 [10 points]

a) $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} -1 & 4 \\ 4 & 1 \end{pmatrix} = \begin{pmatrix} -3 & 4 \\ 12 & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -3 & 4 \\ 12 & 1 \end{pmatrix} \frac{-1}{17}\begin{pmatrix} 1 & -4 \\ -4 & -1 \end{pmatrix} = \frac{-1}{17}\begin{pmatrix} -19 & 8 \\ 8 & -49 \end{pmatrix}$

b) $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2 & 5 \end{pmatrix} \Rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2 & 5 \end{pmatrix} \frac{1}{5}\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 1 & 2 \\ -8 & 9 \end{pmatrix}$

Question 7 [10 points]

$\mathbf{M} = \begin{pmatrix} 0 & -1 \\ -2 & 1 \end{pmatrix}$ et Aire$= |7\det(\mathbf{M})| = |7(-2)| = 14$

Question 8 [10 points]

$T(x,y) = T(0,2) + kT(1,1) = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} 0 \\ 2 \end{pmatrix} + k\begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 2 \end{pmatrix} + k\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \end{pmatrix} + k\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Question 9 [10 points]

$$\begin{pmatrix} 1 & -\frac{1}{5} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \frac{5}{13} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{5} & 0 \\ 0 & 1 \end{pmatrix}\mathbf{M} = \mathbf{I}$$

$$\mathbf{M} = \begin{pmatrix} 5 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \frac{13}{5} \end{pmatrix}\begin{pmatrix} 1 & \frac{1}{5} \\ 0 & 1 \end{pmatrix}$$

$\begin{pmatrix} 5 & 0 \\ 0 & 1 \end{pmatrix} \rightarrow$ étirement de facteur 5 dans la direction de $\vec{\imath}$

$\begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix} \rightarrow$ cisaillement de facteur 3 dans la direction de $\vec{\jmath}$

$\begin{pmatrix} 1 & 0 \\ 0 & \frac{13}{5} \end{pmatrix} \rightarrow$ étirement de facteur $\frac{13}{5}$ dans la direction de $\vec{\jmath}$

$\begin{pmatrix} 1 & \frac{1}{5} \\ 0 & 1 \end{pmatrix} \rightarrow$ cisaillement de facteur $\frac{1}{5}$ dans la direction de $\vec{\imath}$

Question 10 [10 points]

$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} 5-\lambda & 2 \\ 8 & -1-\lambda \end{vmatrix} = (5-\lambda)(-1-\lambda) - 16 = \lambda^2 - 4\lambda - 21 \stackrel{?}{=} 0 \Rightarrow \lambda = \frac{4 \pm \sqrt{16+84}}{2} = 7, -3$

Pour $\lambda_1 = 7$, $\begin{pmatrix} -2 & 2 \\ 8 & -8 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \vec{u} = (1,1)$

Pour $\lambda_2 = -3$, $\begin{pmatrix} 8 & 2 \\ 8 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \vec{v} = (1,-4)$